# The functor $beta_{Y}(cdot)$ and mixed problems for $mathcal{D}_{X}$-modules (Microlocal Analysis and Singular Perturbation Theory)



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# The functor $\beta_Y(\cdot)$ and mixed problems for $\mathscr{D}_X$-modules

*Dedicated to Professor H. Komatsu on his eighty years birthday, and to Professor T. Kawai on his seventy years birthday*

By

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## Abstract

Let $M$ be a real analytic manifold, and $N$ be its real analytic submanifold with codimension 1. We denote by $X, Y$ their complexifications. First, we give an algebraic formulation of mixed initial boundary value problems for coherent left $\mathscr{D}_X$-modules by using sheaf $\beta_Y(\mathscr{O}_X)$ introduced in [1]. This formulation is of coordinate free because $\beta_Y(\mathscr{O}_X)$ is defined only on $X$ and $Y$. At the same time, we give a functorial construction of $\beta_Y(\mathscr{O}_X)$. The main results under this formulation are coordinate-free generalizations of our previous results [2] for single differential equations; for example, the estimate of the micro-support of some important solution complex. We will give in [3] the detailed proofs and applications; the existence results of hyperfunction solutions, and the propagation results of micro-analyticity of the solutions along the boundary $N$ as obtained by J. Sjöstrand [4].

## § 1. A brief introduction of mixed problems for single equations

Let $P(t, x, \partial_t, \partial_x) = \partial_t^2 + A_1(t, x, \partial_x)\partial_t + A_2(t, x, \partial_x)$ be a second order differential operator on $M := \mathbb{R}_t \times \mathbb{R}_x^n$ with $C^\omega$-coefficients, to which the boundary $N := \{x_1 = 0\}$ is non-characteristic. Consider a solution $u(t, x)$ in $\{(t, x) \in \mathbb{R}_t \times \mathbb{R}_x \mid x_1 > 0\}$ to an initial and boundary value problem

$$(1.1) \qquad \begin{cases} P(t, x, \partial_t, \partial_x)u(t, x) & = f(t, x) \quad (x_1 > 0), \\ \qquad u(t, x)|_{x_1=+0} & = g(t, x'), \\ \qquad \partial_t^j u(0, x) & = h_j(x) \quad (j = 0, 1,\ x_1 > 0). \end{cases}$$

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Here, $f, g, h_j$ are given functions; $x' = (x_2, \dots, x_n)$, and $t$ is the time variable. Then it is easy to see that we may take

$$g(t,x) = h_0(x) = h_1(x) = 0$$

without loss of generality. Further by considering $u(t,x)Y(\pm t) =: u_\pm(t,x)$ instead of $u(t,x)$, we get

$$P(t,x,\partial_t,\partial_x)u_\pm(t,x) = f(t,x)Y(\pm t),$$

where $Y(t)$ is the Heaviside function. Since $u(0,x) = \partial_t u(0,x) = 0$, for a $C^2$-class solution $u$ we can reduce the original mixed problem to the following problem for $u_\pm(t,x)$ with $f_\pm := f(t,x)Y(\pm t)$:

$$\text{(1.2)} \qquad \begin{cases} P(t,x,\partial_t,\partial_x)u_\pm(t,x) & = f_\pm(t,x) \quad (x_1 > 0), \\ \quad u(t,x)|_{x_1=+0} & = 0, \\ \quad \mathrm{supp}\,(u_\pm(t,x)) & \subset \{\pm t \geq 0\}. \end{cases}$$

Because $\{x_1 = 0\}$ is non-characteristic to $P$, we can consider the canonical extension $\widetilde{u_\pm}(t,x)$ of $u_\pm(t,x)$ to $x_1 \leq 0$ (formally $\widetilde{u_\pm}(t,x) = u_\pm(t,x)Y(x_1)$). Hence our problem reduces to the following:

$$\begin{cases} P(t,x,\partial_t,\partial_x)\widetilde{u_\pm}(t,x) = \widetilde{f_\pm}(t,x) + a(t,0,x')\dfrac{\partial u_\pm}{\partial x_1}(t,+0,x')\delta(x_1), \\ \quad \mathrm{supp}\,(\widetilde{u_\pm}(t,x)) \subset \{x_1 \geq 0\} \cap \{\pm t \geq 0\}. \end{cases}$$

Here, $a(t,x)$ is the coefficient of $\partial_t^2$ in $P$, and $\widetilde{f_\pm}(t,x)$ is the canonical extension of $f_\pm(t,x)$ to $x_1 \leq 0$. Then, applying $x_1\times$ to the first equation, we get

$$\text{(1.3)} \qquad \begin{cases} x_1 P(t,x,\partial_t,\partial_x)\widetilde{u_\pm}(t,x) = x_1\widetilde{f_\pm}(t,x), \\ \quad \mathrm{supp}\,(\widetilde{u_\pm}(t,x)) \subset \{x_1 \geq 0\} \cap \{\pm t \geq 0\}. \end{cases}$$

Since $\widetilde{u_\pm}(t,x), x_1\widetilde{f_\pm}(t,x) \in \mathit{\Gamma}_{\{x_1 \geq 0, \pm t \geq 0\}}(\mathscr{B}_{\mathbb{R}_t \times \mathbb{R}_x^n})$, this is equivalent to calculating the cohomology groups of the following complex of sheaves:

$$\text{(1.4)} \qquad 0 \longrightarrow \mathit{\Gamma}_{\{x_1 \geq 0, \pm t \geq 0\}}(\mathscr{B}_{\mathbb{R}_t \times \mathbb{R}_x^n}) \xrightarrow{x_1 P} \mathit{\Gamma}_{\{x_1 \geq 0, \pm t \geq 0\}}(\mathscr{B}_{\mathbb{R}_t \times \mathbb{R}_x^n}) \longrightarrow 0.$$

Therefore, the unique solvability of (1.3) at $(0,0,\mathring{x}')$ is equivalent to the exactness of the complex (1.4) at $(0,0,\mathring{x}')$. Thus, we can formulate the mixed initial boundary value problem algebraically. On the other hand, since the sheaf of Sato hyperfunctions is flabby, (1.4) is algebraically derived from the complex $\mathscr{F}$ on $\mathbb{R}_t \times \mathbb{R}_x^n$:

$$\text{(1.5)} \qquad \mathscr{F} : 0 \longrightarrow \mathit{\Gamma}_{\{x_1 \geq 0\}}(\mathscr{B}_{\mathbb{R}_t \times \mathbb{R}_x^n}) \xrightarrow{x_1 P} \mathit{\Gamma}_{\{x_1 \geq 0\}}(\mathscr{B}_{\mathbb{R}_t \times \mathbb{R}_x^n}) \longrightarrow 0$$

after applying the functor $\mathbb{R}\Gamma_{\{\pm t\geq 0\}}(\cdot)$; that is, $\mathbb{R}\Gamma_{\{\pm t\geq 0\}}(\mathscr{F})$. Further, since the exactness is expressed as

$$\mathbb{R}\Gamma_{\{\pm t\geq 0\}}(\mathscr{F})\big|_{(0,0,\mathring{x}')} = 0,$$

we can use the micro-support theory due to Kashiwara-Schapira [5] as

$$(0,0,\mathring{x}'; \pm dt) \notin \mathrm{SS}(\mathscr{F}).$$

Here, $\mathrm{SS}(\mathscr{G})$ (the micro-support of a sheaf complex $\mathscr{G}$) is a conic closed subset of $T^*Z$ for a sheaf complex $\mathscr{G}$ on a $C^1$ manifold $Z$. By using such an expression, we can neglect the initial surface $\{t=0\}$, and so hereafter, we change the notation of variables as $N=\{t=0\}$; that is,

$$M=\mathbb{R}_t\times\mathbb{R}^n_x \supset N=\{(t,x)\in M \mid t=0\}, \quad X=\mathbb{C}_{\tilde{t}}\times\mathbb{C}^n_z \supset Y=\{\tilde{t}=0\}\times\mathbb{C}^n_z,$$

where $X, Y, \tilde{t}, z$ are the complexifications of $M, N, t, x$, respectively. Then, for the differential operator $P(t,x,\partial_t,\partial_x)$ in the real domain, we put a coherent left $\mathscr{D}_X$-module

$$\widetilde{\mathscr{M}} := \mathscr{D}_X \Big/ \big(\mathscr{D}_X\ \tilde{t}{\cdot}P(\tilde{t},z,\partial_{\tilde{t}},\partial_z)\big).$$

Since $\Gamma_{\{t\geq 0\}}(\mathscr{B}_M) = \mathbb{R}\Gamma_M\left(\Gamma_{\{t\geq 0\}}(\mathscr{B}\mathscr{O}_{\mathbb{R}_t\times\mathbb{C}^n_z})\right)[n]$, we have

$$\begin{aligned}\mathscr{F} &= \mathbb{R}\mathscr{H}om_{\mathscr{D}_X}(\widetilde{\mathscr{M}}, \Gamma_{\{t\geq 0\}}(\mathscr{B}_M)) \\ &= \mathbb{R}\Gamma_M\left(\mathbb{R}\mathscr{H}om_{\mathscr{D}_X}(\widetilde{\mathscr{M}}, \Gamma_{\{t\geq 0\}}(\mathscr{B}\mathscr{O}_{\mathbb{R}_t\times\mathbb{C}^n_z}))\right)[n],\end{aligned}$$

where $\mathscr{B}\mathscr{O}_{\mathbb{R}_t\times\mathbb{C}^n_z}$ is the sheaf of hyperfunctions of $t, \operatorname{Re} z, \operatorname{Im} z$ depending holomorphically in $z$. Thus, we obtain a more fundamental sheaf complex $\mathscr{G}$ on $\mathbb{R}_t\times\mathbb{C}^n_z$:

$$\mathscr{G} := \mathbb{R}\mathscr{H}om_{\mathscr{D}_X}(\widetilde{\mathscr{M}}, \Gamma_{\{t\geq 0\}}(\mathscr{B}\mathscr{O}_{\mathbb{R}_t\times\mathbb{C}^n_z})).$$

Indeed, Kashiwara-Schapira's micro-support theory gives a geometric estimate of $\mathrm{SS}(\mathbb{R}\Gamma_M(\mathscr{G}))$ by $\mathrm{SS}(\mathscr{G})$. So we can bypass the most difficult analytic arguments for mixed problems by using micro-support theory [5]. Further it is easy to see that the cohomology groups of the complex

$$\mathscr{G}: 0 \longrightarrow \Gamma_{\{t\geq 0\}}(\mathscr{B}\mathscr{O}_{\mathbb{R}_t\times\mathbb{C}^n_z}) \xrightarrow{t\cdot P} \Gamma_{\{t\geq 0\}}(\mathscr{B}\mathscr{O}_{\mathbb{R}_t\times\mathbb{C}^n_z}) \longrightarrow 0 \tag{1.6}$$

is concentrated in degree 0. Indeed, since $\{t=0\}$ is non-characteristic to $P(t,z,\partial_t,\partial_z)$, $t\cdot P$ is a surjective morphism. Consequently we have the following quasi-isomorphism:

$$\mathscr{G} \simeq \ker(tP) \simeq \mathscr{K},$$

where

$$\mathscr{K} := \{U(t,z)\in \mathscr{O}_X|_{[0,+\infty)_t\times\mathbb{C}^n_z} \mid PU=0, U(0,z)=0\}, \tag{1.7}$$

$$\mathscr{K} \ni U(t,z) \longmapsto U(t,z)Y(t) \in \ker(tP). \tag{1.8}$$

The following is the simplest case in [2] for the Dirichlet boundary condition:

$$U(0,z) = \partial_{\tilde{t}} U(0,z) = \cdots = \partial_{\tilde{t}}^{k-1} U(0,z) = 0.$$

**Theorem 1.1.** *Let $P(\tilde{t}, z, \partial_{\tilde{t}}, \partial_z) = \partial_{\tilde{t}}^m + A_1(\tilde{t}, z, \partial_z)\partial_{\tilde{t}}^{m-1} + \cdots + A_m(\tilde{t}, z, \partial_z)$ be an $m$-th order holomorphic differential operator, and let $k$ be an integer such that $1 \leq k \leq m-1$. Put*

$$\widetilde{\mathscr{M}} := \mathscr{D}_X \Big/ \big(\mathscr{D}_X \, \tilde{t}^{\,m-k} \cdot P(\tilde{t}, z, \partial_{\tilde{t}}, \partial_z)\big).$$

*For any $\mathring{t} \geq 0$, $\mathring{\zeta} \neq 0$, set $p = (\mathring{t}, \mathring{z}; \mathring{\tau} dt + \mathrm{Re}(\mathring{\zeta} dz)) \in T^*(\mathbb{R}_t \times \mathbb{C}_z^n)$. Then,*

$$p \notin \mathrm{SS}(\mathbb{R}\mathscr{H}om_{\mathscr{D}_X}(\widetilde{\mathscr{M}}, \mathit{\Gamma}_{\{t \geq 0\}}(\mathscr{B}\mathscr{O}_{\mathbb{R}_t \times \mathbb{C}_z^n})))$$

*holds if the following (1) or (2) is satisfied:*

(1) $\mathring{t} > 0$, *and* $\sigma(P)(\mathring{t}, \mathring{z}, w + \mathring{\tau}, \mathring{\zeta}) \neq 0 \ \ (\forall w \in \sqrt{-1}\mathbb{R})$.

(2) $\mathring{t} = 0$, $\sigma(P)(0, \mathring{z}, w + \mathring{\tau}, \mathring{\zeta}) \neq 0 \ \ (\forall w \in \sqrt{-1}\mathbb{R})$, *and*

$$k = \#\{w \in \mathbb{C} \mid \sigma(P)(0, \mathring{z}, w + \mathring{\tau}, \mathring{\zeta}) = 0,\ \mathrm{Re}\, w < 0\}.$$

*Remark.*

- This theorem is extended in [2] to the case with general $k$ boundary conditions, where we need Lopatinskii conditions.
- The case (1) is just the condition due to Kashiwara-Kawai [6].
- A. Martinez [7] obtained the condition (2) for the analytic extension of holomorphic solutions satisfying boundary conditions across a surface. Our result includes not only such a result, but also extensions of higher cohomology groups across the surface.

## § 2. Sheaf $\beta_Y(\mathscr{O}_X)$

It is clear that the sheaf $\mathit{\Gamma}_{\{t\geq 0\}}(\mathscr{B}\mathscr{O}_{\mathbb{R}_t \times \mathbb{C}_z^n})$ used in [2] depends on the product structure of $X = \mathbb{C}_{\tilde{t}} \times \mathbb{C}_z^n$. On the other hand, the solution complex at (1.7) has a natural extension to the real monoidal transformation

$$\widetilde{{}^Y X} := (X \setminus Y) \sqcup S_Y X$$

of $X$ with center $Y$. So it is natural to extend $\Gamma_{\{t\geq 0\}}(\mathscr{B}\mathscr{O}_{\mathbb{R}_t\times\mathbb{C}^n_z})$ to $\widetilde{{}^Y X}$. Let $M$ be a real analytic manifold, and $N$ be its submanifols with codimension 1. Let $X\supset Y$ be their complexifications. We use a local coordinate system $(t,x_1,\dots,x_n)$ of $M$ such that $N=\{t=0\}$. For simplicities, we set $M=\mathbb{R}^{n+1}, N=\{(t,x)\in M \mid t=0\}$. Further, let $\tilde t, z_1,\dots,z_n$ be the complexifications of $t,x_1,\dots,x_n$. Then by using the polar coordinates $(r,\theta)$ for $\tilde t = re^{i\theta}$, the real monoidal transformation $\widetilde{{}^Y X}:=(X\setminus Y)\sqcup S_Y X$ of $X$ with center $Y$ is defined as a real analytic manifold with boundary:

$$\begin{aligned}\widetilde{{}^Y X}=\{(r,\theta,x,y)\in Z; r\geq 0\}\hookrightarrow Z:=\mathbb{R}_r\times(\mathbb{R}/2\pi\mathbb{Z})_\theta\times\mathbb{R}^{2n}_{x,y}\\ \xrightarrow{\tau}(re^{i\theta},x+iy)\in X,\end{aligned}\tag{2.1}$$

where $\tau: Z\to X$ is a natural map. We introduced a sheaf $\beta_Y(\mathscr{O}_X)$ on $Z$ in [1] such that the support of $\beta_Y(\mathscr{O}_X)$ is $\widetilde{{}^Y X}$ and that it satisfies the exact sequence:

$$0\longrightarrow \tau^{-1}\mathscr{H}^1_Y(\mathscr{O}_X)\longrightarrow\beta_Y(\mathscr{O}_X)\longrightarrow j_*(\mathscr{O}_X|_{X\setminus Y})\longrightarrow 0,\tag{2.2}$$

where $j$ is a natural map:

$$X\setminus Y\simeq(0,\infty)_r\times(\mathbb{R}/2\pi\mathbb{Z})_\theta\times\mathbb{R}^{2n}_{x,y}\xrightarrow{j}\widetilde{{}^Y X}.$$

We have three different definitions as follows:

**1. Stalkwise definition of $\beta_Y(\mathscr{O}_X)$** : The stalk at $p=(\mathring{r},\mathring{\theta},\mathring{z})$ is

$$\beta_Y(\mathscr{O}_X)\Big|_{(\mathring{r},\mathring{\theta},\mathring{z})}=\begin{cases}\mathscr{O}_X\big|_{\tau(p)} & (\mathring{r}>0),\\ \varinjlim_{\delta\to+0}\mathscr{O}_{\widetilde{X\setminus Y}}\left(U_\delta(\mathring{\theta},\mathring{z})\right)\Big/\mathscr{O}|_{\tau(p)} & (\mathring{r}=0),\\ 0 & (\mathring{r}<0),\end{cases}\tag{2.3}$$

where $\widetilde{X\setminus Y}$ is the universal covering of $X\setminus Y$, and (see Figure 1)

$$U_\delta(\mathring{\theta},\mathring{z}):=\{(re^{i\theta},z)\in\widetilde{X\setminus Y}\mid 0<r<\delta,\ -\delta<\theta-\mathring{\theta}<2\pi+\delta,\ |z-\mathring{z}|<\delta\}.$$

**2. Analytic definition of $\beta_Y(\mathscr{O}_X)$** : A hyperfunction $u(r,\theta,x,y)$ of $r,\theta,x,y$ satisfying the following conditions:

$$\begin{cases}\operatorname{supp}(u)\subset\{r\geq 0\},\\ (r\partial_r+i\partial_\theta)u=0,\ (\partial_{x_k}+i\partial_{y_k})u=0\ (k=1,\dots,n).\end{cases}\tag{2.4}$$

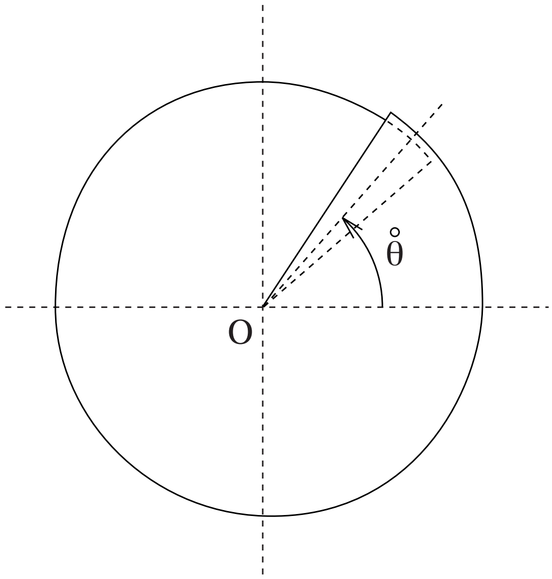


Figure 1. $U_\delta$

**3. Functorial definition of $\beta_Y(\mathscr{O}_X)$:** We can construct a functor $\beta_Y(\cdot)$:

$$\beta_Y : D^+(X) \longrightarrow D^+(\widetilde{^Y X}), \tag{2.5}$$

satisfying a distinguished triangle for any $\mathscr{F} \in D^+(X)$:

$$\tau^{-1}\mathbb{R}\Gamma_Y(\mathscr{F})[1] \to \beta_Y(\mathscr{F}) \to \mathbb{R}j_*(\mathscr{F}|_{X\setminus Y}) \xrightarrow{+1} . \tag{2.6}$$

where

$$X \setminus Y \overset{j}{\hookrightarrow} \widetilde{^Y X} = (X \setminus Y) \sqcup S_Y X \subset Z \xrightarrow{\tau} X.$$

$\widetilde{^Y X}_{\mathrm{dc}}$ : The double covering space of $\widetilde{^Y X}$. That is,

$$\begin{aligned} \widetilde{^Y X}_{\mathrm{dc}} &\simeq [0,+\infty)_r \times (\mathbb{R}/4\pi\mathbb{Z})_\theta \times \mathbb{C}^n_z \\ &\xrightarrow{p_{\mathrm{dc}}} [0,+\infty)_r \times (\mathbb{R}/2\pi\mathbb{Z})_\theta \times \mathbb{C}^n_z \simeq \widetilde{^Y X}, \end{aligned} \tag{2.7}$$

where $p_{\mathrm{dc}}$ is the covering map. Let $\widetilde{^Y X}^{(j)}_{\mathrm{dc}}$ $(j = 1, 2)$ be two copies of $\widetilde{^Y X}_{\mathrm{dc}}$, and make a fiber product of them concerning

$$\tau \circ p^{(j)}_{\mathrm{dc}} : \widetilde{^Y X}^{(j)}_{\mathrm{dc}} \to X \quad (j = 1, 2).$$

Namely we have

$$\widetilde{^Y X}^{(1)}_{\mathrm{dc}} \underset{X}{\times} \widetilde{^Y X}^{(2)}_{\mathrm{dc}} \simeq \{(r, \theta_1, \theta_2, z) \in [0,+\infty) \times (\mathbb{R}/4\pi\mathbb{Z})^2 \times \mathbb{C}^n;\ re^{i\theta_1} = re^{i\theta_2}\}. \tag{2.8}$$

Let us define a closed subset $\widetilde{{}^Y X}_\beta$ of $\widetilde{{}^Y X}^{(1)}_{\mathrm{dc}} \underset{X}{\times} \widetilde{{}^Y X}^{(2)}_{\mathrm{dc}}$:

$$
\begin{aligned}
(2.9) \qquad \widetilde{{}^Y X}_\beta &= \{(r,\theta_1,\theta_2,z) \in \widetilde{{}^Y X}^{(1)}_{\mathrm{dc}} \underset{X}{\times} \widetilde{{}^Y X}^{(2)}_{\mathrm{dc}} \ ;\ \sin\left(\frac{\theta_1-\theta_2}{2}\right) \geq 0\} \\
&\simeq \{0\}_r \times \{(\theta+[\varphi],\theta) \in (\mathbb{R}/4\pi\mathbb{Z})^2; \theta \in \mathbb{R}/4\pi\mathbb{Z}, 0 \leq \varphi \leq 2\pi\} \times \mathbb{C}^n_z \\
&\cup [0,\infty)_r \times \{(\theta+[\varphi],\theta) \in (\mathbb{R}/4\pi\mathbb{Z})^2; \theta \in \mathbb{R}/4\pi\mathbb{Z}, \varphi = 0, 2\pi\} \times \mathbb{C}^n_z.
\end{aligned}
$$

$\widetilde{{}^Y X}_\beta$ does not depend on the choice of coordinate systems $(r,\theta,x+iy)$ of $\widetilde{{}^Y X}$. Let $\nu_j$ be $p^{(j)}_{\mathrm{dc}}$ the natural projections:

$$
(2.10) \qquad \widetilde{{}^Y X}_\beta \xrightarrow{\nu_j} \widetilde{{}^Y X}^{(j)}_{\mathrm{dc}} \xrightarrow{p^{(j)}_{\mathrm{dc}}} \widetilde{{}^Y X}.
$$

Then, for an $\mathscr{F} \in D^+(X)$, we have a natural morphism $\varepsilon$:

$$
\varepsilon : \big(p^{(2)}_{\mathrm{dc}}\big)^{-1}\tau^{-1}\mathscr{F} \longrightarrow \mathbb{R}\nu_{2*}\nu_1^{-1}\big(p^{(1)}_{\mathrm{dc}}\big)^{-1}\mathbb{R}\Gamma_{X\setminus Y}(\tau^{-1}\mathscr{F}).
$$

By using this $\varepsilon$, we define $\beta'_Y(\cdot)$ as

$$
\beta'_Y(\cdot) := M(\varepsilon) \text{ (a mapping cone)} : D^+(X) \longrightarrow D^+(\widetilde{{}^Y X}^{(2)}_{\mathrm{dc}}).
$$

Since $\beta'_Y(\cdot)$ is invariable under covering transformations, $\beta'_Y(\cdot)$ induces

$$
\beta_Y(\cdot) : D^+(X) \longrightarrow D^+(\widetilde{{}^Y X}).
$$

Therefore we obtain the triangle (2.6), and the fundamental exact sequence (2.1) for $\mathscr{F} = \mathscr{O}_X$ by using the following proposition.

**Proposition 2.1.** *For a sheaf $\mathscr{G}$ on $\widetilde{{}^Y X}$, we have an exact sequence of sheaves on $\widetilde{{}^Y X}^{(2)}_{dc}$:*

$$
(2.11) \qquad 0 \longrightarrow \big(p^{(2)}_{dc}\big)^{-1}\tau^{-1}\tau_*\mathscr{G} \xrightarrow{\lambda} \nu_{2*}\nu_1^{-1}\big(p^{(1)}_{dc}\big)^{-1}\mathscr{G} \xrightarrow{\kappa} \big(p^{(2)}_{dc}\big)^{-1}\mathscr{G}.
$$

*Further, if $\mathscr{G}$ is flabby on $\widetilde{{}^Y X}$, then $\kappa$ is surjective.*

## § 3. A formulation of mixed problems for $\mathscr{D}_X$-modules

We generalize the module

$$
\mathscr{M} = \mathscr{D}_X \Big/ \big(\mathscr{D}_X\, \tilde{t}^{\,m-k} \cdot P(\tilde{t}, z, \partial_t, \partial_z)\big)
$$

for a single operator $P(t,x,\partial_t,\partial_x)$ with $k$-Dirichlet conditions.

**Definition 3.1. (A triple for mixed problems)**: $(\mathscr{M}, \mathscr{N}, \varphi)$ is said to a triple for mixed problems concerning the pair $M \supset N$ if they satisfy the following (i)~(iii).

(i) $\mathscr{M}$ is a coherent left $\mathscr{D}_X$-module, and $Y$ is non-characteristic to $\mathscr{M}$.

(ii) $\mathscr{N}$ is a coherent left $\mathscr{D}_Y$-module.

(iii) $\varphi : \mathscr{M}_Y = \mathscr{M}/(\tilde{t}\cdot\mathscr{M}) \longrightarrow \mathscr{N}$ is a surjective $\mathscr{D}_Y$-morphism.

Then we define a $\mathscr{D}_Y$-module $\mathscr{N}^{\perp}$ as

$$\mathscr{N}^{\perp} := \ker\varphi. \tag{3.1}$$

Easily to see, it is equivalent to give a $\mathscr{D}_Y$-submodule $\mathscr{N}^{\perp}$ of $\mathscr{M}_Y$ instead of $\varphi$.

**Theorem 3.2. (Construction of $\widetilde{\mathscr{M}_{\varphi}}$).** *Let $(\mathscr{M}, \mathscr{N}, \varphi)$ be a triple for mixed problems concerning $M \supset N$. Then we have a unique $\mathscr{D}_X$-module $\widetilde{\mathscr{M}_{\varphi}}$ satisfying all the following conditions:*

*(i) $\widetilde{\mathscr{M}_{\varphi}}$ satisfies the following $\mathscr{D}_X$-exact sequence:*

$$0 \longrightarrow \mathscr{D}_{X\leftarrow Y} \underset{\mathscr{D}_Y}{\otimes} \mathscr{N} \xrightarrow{\iota} \widetilde{\mathscr{M}_{\varphi}} \xrightarrow{\rho} \mathscr{M} \longrightarrow 0. \tag{3.2}$$

*(ii) There exists an $\mathscr{O}_X$-linear morphism $\mathrm{ext} : \mathscr{M} \to \widetilde{\mathscr{M}_{\varphi}}$ such that $\rho\cdot\mathrm{ext} = \mathrm{id}$ and that: For any $V \in \mathscr{M}$, and any $P \in \mathscr{D}_X$ satisfying $[P, \tilde{t}] = 0$ we have*

$$\begin{cases} P(\tilde{t}, z, \partial_z)\mathrm{ext}(V) = \mathrm{ext}(P(\tilde{t}, z, \partial_z)V), \\ \partial_{\tilde{t}}\,\mathrm{ext}(V) = \mathrm{ext}(\partial_{\tilde{t}}\, V) + \iota(\delta(\tilde{t}) \otimes \varphi([V])), \end{cases} \tag{3.3}$$

*where $\delta(\tilde{t}) = \delta(\tilde{t})\delta(z - z')dz'$. Further $[V]$ means the equivalence class in $\mathscr{M}_Y$ for $V \in \mathscr{M}$.*

**Definition 3.3. ( Lopatinskii conditions).** Let $\tau^*, \iota^*$ be the following maps:

$$\begin{aligned} \tau^* : T^*X \underset{X}{\times} Z \ni \big(r, \theta, x, y; \mathrm{Re}(w d\tilde{t} + (\xi + i\eta)dz)\big) \\ \longmapsto \big(r, \theta, x, y; \mathrm{Re}(we^{i\theta}dr + ire^{i\theta}wd\theta) + \xi dx - \eta dy\big) \in T^*Z, \end{aligned} \tag{3.4}$$

$$\iota^* : T^*X \underset{X}{\times} Y \longrightarrow T^*Y. \tag{3.5}$$

Then we say that a triple $(\mathscr{M}, \mathscr{N}, \varphi)$ for mixed problems concerning $M \supset N$ satisfies the Lopatinskii condition at

$$p = (0, \mathring{\theta}, \mathring{x}, \mathring{y}; \mathring{\mu}dr - \mathring{\nu}d\theta + \mathring{\xi}dx - \mathring{\eta}dy) \in T^*Z \cap \{r = \nu = 0, \xi + i\eta \neq 0\}$$

if all the following conditions hold:

(*i*) $\mathrm{char}(\mathscr{M}) \cap (\tau^*)^{-1}(p) = \emptyset$, where

$$(\tau^*)^{-1}(p) = \{(0, \mathring{x} + i\mathring{y}; \mathrm{Re}(e^{-i\mathring{\theta}}(\mathring{\mu} + i\lambda)d\widetilde{t} + (\mathring{\xi} + i\mathring{\eta})dz) \mid \lambda \in \mathbb{R}\}.$$

(*ii*) Put $\mathscr{M}^* := \mathscr{E}_X \underset{\pi^{-1}\mathscr{D}_X}{\otimes} \pi^{-1}\mathscr{M}$. Then, under (*i*) we can decompose $\mathscr{M}^*$ as

$$\begin{aligned}&\mathscr{M}^* = \mathscr{M}^{*,+} \oplus \mathscr{M}^{*,-},\\ &\mathrm{supp}(\mathscr{M}^{*,\pm}) \cap (\iota^*)^{-1}((\mathring{z}, \mathring{\zeta})) \subset \{w \in \mathbb{C} \mid \pm\mathrm{Re}(we^{i\mathring{\theta}} - \mathring{\mu}) > 0\}.\end{aligned}$$

in a neighborhood of

$$(\iota^*)^{-1}((\mathring{z}, \mathring{\zeta})) = \{(0, \mathring{x} + i\mathring{y}; \mathrm{Re}(wd\widetilde{t} + (\mathring{\xi} + i\mathring{\eta})dz) \mid w \in \mathbb{C}\}.$$

Then we suppose that the following $\mathscr{E}_Y$-morphism is isomorphic:

$$\varphi^*|_{\mathscr{M}_Y^{*,+}} : (\iota^*)_*\big(\mathscr{M}^{*,+}/(\tilde{t}\cdot\mathscr{M}^{*,+})\big) \xrightarrow{\sim} \mathscr{E}_Y \underset{\pi^{-1}\mathscr{D}_Y}{\otimes} \mathscr{N}. \tag{3.6}$$

or equivalently we have

$$(\iota^*)_*\big(\mathscr{M}^{*,-}/(\tilde{t}\cdot\mathscr{M}^{*,-})\big) = \mathscr{E}_Y \underset{\pi^{-1}\mathscr{D}_Y}{\otimes} \mathscr{N}^{\perp}. \tag{3.7}$$

**Theorem 3.4.** **(The microsupport of $\mathbb{R}\mathscr{H}om_{\tau^{-1}\mathscr{D}_X}(\tau^{-1}\widetilde{\mathscr{M}_\varphi}; \beta_Y(\mathscr{O}_X))$).** *Let $(\mathscr{M}, \mathscr{N}, \varphi)$ be a triple for mixed problems concerning $M \supset N$, and $\widetilde{\mathscr{M}_\varphi}$ be the $\mathscr{D}_X$-module constructed in Theorem 3.2. Then for a point*

$$p = (\mathring{r}, \mathring{\theta}, ; \mathring{\mu}dr - \mathring{\nu}d\theta + \mathring{\xi}dx - \mathring{\eta}dy) \in T^*Z \cap \{r \geq 0, \xi + i\eta \neq 0\},$$

*we have*

$$p \notin \mathrm{SS}\big(\mathbb{R}\mathscr{H}om_{\tau^{-1}\mathscr{D}_X}(\tau^{-1}\widetilde{\mathscr{M}_\varphi}; \beta_Y(\mathscr{O}_X))\big),$$

*if one of the following conditions is satisfied:*

(*i*) $\mathring{r} > 0$ *and*

$$\left(\mathring{r}e^{i\mathring{\theta}}, \mathring{x} + i\mathring{y}; e^{-i\mathring{\theta}}(\mathring{\mu} + i\,\frac{\mathring{\nu}}{\mathring{r}})d\tilde{t} + (\mathring{\xi} + i\mathring{\eta})dz\right) \notin \mathrm{char}(\mathscr{M}).$$

(*ii*) $\mathring{r} = 0,\ \mathring{\nu} \neq 0.$

(*iii*) $\mathring{r} = 0,\ \mathring{\nu} = 0,$ *and the Lopatinskii condition is satisfied at* $p$.

**Example 3.5. (Single differential equations).**
$P(\tilde{t}, z, \partial_{\tilde{t}}, \partial_z) = \partial_{\tilde{t}}^m + \sum_{q=1}^m A_q(\tilde{t}, z, \partial_z)\partial_{\tilde{t}}^{m-q}$: order $m$ PDE,
$B_j(z, \partial_{\tilde{t}}, \partial_z) = \sum_{q=0}^{m-1} B_{jq}(z, \partial_z)\partial_{\tilde{t}}^q$ F order $\ell_j$ $C^\omega$-PDE's $(j = 1, \dots, k)$.

$$\mathscr{M} := \mathscr{D}_X/(\mathscr{D}_X P), \quad \mathscr{N} := (\mathscr{D}_Y)_{q=0}^{m-1}/(\sum_{j=1}^k \mathscr{D}_Y B_{jq})_{q=0}^{m-1},$$

$$\varphi : \mathscr{M}_Y \ni \sum_{q=0}^{m-1} R_q(z, \partial_z)\partial_{\tilde{t}}^q \longmapsto \Big[\big(R_q(z, \partial_z)\big)_{q=0}^{m-1}\Big] \in \mathscr{N},$$

Then, the Lopatinskii condition at $p = (0, \mathring{\theta}, \mathring{x}, \mathring{y}; \mathring{\mu}dr - 0d\theta + \mathring{\xi}dx - \mathring{\eta}dy) \in T^*Z$ is given as follows:

(1) Assume $\sigma(P)(0, \mathring{x} + i\mathring{y}; e^{-i\mathring{\theta}}(w + \mu_0), \mathring{\xi} + i\mathring{\eta}) \neq 0$ $(\forall w \in \sqrt{-1}\mathbb{R})$, and put

$$r = \#\{w \in \mathbb{C} \mid \sigma(P)(0, \mathring{x} + i\mathring{y}; e^{-i\mathring{\theta}}(w + \mu_0), \mathring{\xi} + i\mathring{\eta}) = 0,\ \operatorname{Re} w < 0\}.$$

Then decompose

$$P(\tilde{t}, z, \partial_{\tilde{t}}, \partial_z) = P_+(\tilde{t}, z, \partial_{\tilde{t}}, \partial_z) \cdot P_-(\tilde{t}, z, \partial_{\tilde{t}}, \partial_z),$$

where

$$\{w \in \mathbb{C} \mid \sigma(P_\pm)(0, \mathring{x} + i\mathring{y}; e^{-i\mathring{\theta}}(w + \mu_0), \mathring{\xi} + i\mathring{\eta}) = 0\} \subset \{\pm \operatorname{Re} w > 0\}.$$

Therefore, putting $P_- = \partial_{\tilde{t}}^r + \sum_{q=0}^{r-1} C_q(\tilde{t}, z, \partial_z)\partial_{\tilde{t}}^q$, we have

$$\mathscr{M}^{*,-} = \mathscr{E}_X/\mathscr{E}_X P_-, \quad (\mathscr{M}^{*,-})_Y = (\mathscr{E}_Y)_{q=0}^{r-1}.$$

On the other hand, the injective morphism $\mathscr{M}^{*,-} \to \mathscr{M}^*$ is given by

$$\mathscr{M}^{*,-} \ni \sum_{q=0}^{r-1} R_q(\tilde{t}, z, \partial_z)\partial_{\tilde{t}}^q \mapsto [\sum_{q=0}^{r-1} R_q \cdot \{\partial_{\tilde{t}}^q + U_q^-(\tilde{t}, z, \partial_{\tilde{t}}, \partial_z)P_-\}] \in \mathscr{M}^*.$$

Here, microdifferential operators $U_q^\pm(\tilde{t}, z, \partial_{\tilde{t}}, \partial_z)$ are defined in

$$\{(0, \mathring{x} + i\mathring{y}; e^{-i\mathring{\theta}}(w + \mu_0), \mathring{\xi} + i\mathring{\eta}); w \in \mathbb{C}, \pm \operatorname{Re} w \geq 0\},$$

and satisfy the following equations:

$$\partial_{\tilde{t}}^q = -U_q^-(\tilde{t}, z, \partial_{\tilde{t}}, \partial_z)P_- + U_q^+(\tilde{t}, z, \partial_{\tilde{t}}, \partial_z)Q_+ \tag{3.8}$$

on $\{(0, \mathring{x} + i\mathring{y}; e^{-i\mathring{\theta}}(w + \mu_0), \mathring{\xi} + i\mathring{\eta}); w \in \mathbb{C}, \operatorname{Re} w = 0\}$, where $Q_\pm$ are microdifferential operators similar to $P_\pm$ except that

$$P(\tilde{t}, z, \partial_{\tilde{t}}, \partial_z) = Q_-(\tilde{t}, z, \partial_{\tilde{t}}, \partial_z) \cdot Q_+(\tilde{t}, z, \partial_{\tilde{t}}, \partial_z).$$

Decompose

$$B_j(z,\partial_{\tilde t},\partial_z)=W_j(\tilde t,z,\partial_{\tilde t},\partial_z)P_-+\sum_{q=0}^{r-1}B'_{jq}(\tilde t,z,\partial_z)\partial_{\tilde t}^q,$$

then condition (3.7) is the following:

$$\begin{aligned}&\left\{\sum_{q=0}^{r-1}R_q(z,\partial_z)\big(\partial_{\tilde t}^q+U_q^-(\tilde t,z,\partial_{\tilde t},\partial_z)P_-\big)\big|_{\tilde t=0};R_q\in\mathscr{E}_Y\right\}\\&=\left\{\sum_{j=1}^{k}S_j(z,\partial_z)\big(W_j(\tilde t,z,\partial_{\tilde t},\partial_z)P_-+\sum_{q=0}^{r-1}B'_{jq}(\tilde t,z,\partial_z)\partial_{\tilde t}^q\big)\big|_{\tilde t=0};S_j\in\mathcal{E}_Y\right\}.\end{aligned}$$

Namely,

$$R_q(z,\partial_z)=\sum_{j=1}^{k}S_j(z,\partial_z)B'_{jq}(0,z,\partial_z).$$

Hence the condition on the isomorphism of $\varphi^*|_{\mathscr{M}^{*,+}}$ is

(1) $r=m-k$; that is, the rank $r$ of $\mathscr{M}_Y^{*,+}$ coincides with that of $\mathscr{N}$. Namely,

$$\begin{aligned}&k(=\text{the number of boundary conditions})\\&=\#\{w\in\mathbb{C}\mid\sigma(P)(0,\mathring{x}+i\mathring{y};e^{-i\mathring{\theta}}(w+\mu_0),\mathring{\xi}+i\mathring{\eta})=0,\ \operatorname{Re}w<0\}.\end{aligned}$$

(2) the matrix $\Big(B'_{j,\ell-1}(0,z,\partial_z)\Big)_{j,\ell=1,\dots,k}$ of microdifferential operators is invertible at $(\mathring{x}+i\mathring{y};\mathring{\xi}+i\mathring{\eta})$.

**Example 3.6. (Applications).**

By applying the functor $\mathbb{R}\mathscr{H}om_{\tau^{-1}\mathscr{D}_X}\big(\tau^{-1}\widetilde{\mathscr{M}_\varphi};\cdot\big)$ to the distinguished triangle

$$\tau^{-1}\mathbb{R}\Gamma_Y(\mathscr{O}_X)[1]\to\beta_Y(\mathscr{O}_X)\to\mathbb{R}j_*(\mathscr{O}_X|_{X\setminus Y})\xrightarrow{+1},$$

we get a distinguished triangle

$$\begin{aligned}&\tau^{-1}\mathbb{R}\mathscr{H}om_{\mathscr{D}_X}\big(\widetilde{\mathscr{M}_\varphi};\mathbb{R}\Gamma_Y(\mathscr{O}_X)\big)[1]\longrightarrow\\&\mathbb{R}\mathscr{H}om_{\tau^{-1}\mathscr{D}_X}\big(\tau^{-1}\widetilde{\mathscr{M}_\varphi};\beta_Y(\mathscr{O}_X)\big)\longrightarrow\\&\mathbb{R}\mathscr{H}om_{\tau^{-1}\mathscr{D}_X}\big(\tau^{-1}\widetilde{\mathscr{M}_\varphi};\mathbb{R}j_*(\mathscr{O}_X|_{X\setminus Y})\big)\xrightarrow{+1}.\end{aligned}$$

It is clear that

$$\mathbb{R}\mathscr{H}om_{\tau^{-1}\mathscr{D}_X}\big(\tau^{-1}\widetilde{\mathscr{M}_\varphi};\mathbb{R}j_*(\mathscr{O}_X|_{X\setminus Y})\big)=\mathbb{R}j_*\big(\tau^{-1}\mathbb{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M};\mathscr{O}_X)\big|_{X\setminus Y}\big).$$

On the other hand, we have

$$\mathbb{R}\mathscr{H}om_{\mathscr{D}_X}\big(\widetilde{\mathscr{M}_\varphi};\mathbb{R}\Gamma_Y(\mathscr{O}_X)\big)[1]=\mathbb{R}\mathscr{H}om_{\mathscr{D}_Y}(\mathscr{N}^\perp;\mathscr{O}_Y).$$

Hence we get the main distinguished triangle

$$\begin{aligned}&\mathbb{R}\mathscr{H}om_{\tau^{-1}\mathscr{D}_X}\big(\tau^{-1}\widetilde{\mathscr{M}_\varphi};\beta_Y(\mathscr{O}_X)\big)\longrightarrow\\&\mathbb{R}j_*\big(\tau^{-1}\mathbb{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M};\mathscr{O}_X)\big|_{X\setminus Y}\big)\to\tau^{-1}\mathbb{R}\mathscr{H}om_{\mathscr{D}_Y}(\mathscr{N}^\perp;\mathscr{O}_Y)\xrightarrow{+1}.\end{aligned}$$

Further, by applying any suitable functors, for example $F:=\mathbb{R}\Gamma_S,\mu_H,\dots$ to this distinguished triangle, we get a new distinguished triangle

$$\begin{aligned}&F\mathbb{R}\mathscr{H}om_{\tau^{-1}\mathscr{D}_X}\big(\tau^{-1}\widetilde{\mathscr{M}_\varphi};\beta_Y(\mathscr{O}_X)\big)\longrightarrow\\&F\mathbb{R}j_*\big(\tau^{-1}\mathbb{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M};\mathscr{O}_X)\big|_{X\setminus Y}\big)\to F\tau^{-1}\mathbb{R}\mathscr{H}om_{\mathscr{D}_Y}(\mathscr{N}^\perp;\mathscr{O}_Y)\xrightarrow{+1}.\end{aligned}$$

Then, if $F(\cdot)$ is a sheaf complex on a manifold $L\subset\mathbb{R}^\ell_u$ and

$p=(\mathring{u};\mathring{\xi}du)\notin \mathrm{SS}\left(F\mathbb{R}\mathscr{H}om_{\tau^{-1}\mathscr{D}_X}(\tau^{-1}\widetilde{\mathscr{M}_\varphi};\beta_Y(\mathscr{O}_X))\right)$, we get some isomorphisms between the related cohomology groups:

$$\begin{aligned}&H^q\left(\mathbb{R}\Gamma_{\{\mathring{\xi}\cdot(u-\mathring{u})\geq 0\}}F\mathbb{R}j_*\big(\tau^{-1}\mathbb{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M};\mathscr{O}_X)\big|_{X\setminus Y}\big)\right)\Big|_{u=\mathring{u}}\\&\quad\xrightarrow{\sim} H^q\left(\mathbb{R}\Gamma_{\{\mathring{\xi}\cdot(u-\mathring{u})\geq 0\}}F\tau^{-1}\mathbb{R}\mathscr{H}om_{\mathscr{D}_Y}(\mathscr{N}^\perp;\mathscr{O}_Y)\right)\Big|_{u=\mathring{u}}\quad(\forall q).\end{aligned}$$

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